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On the Scale of the Nonlinear Effect in a Crack Problem

This note provides a necessary and sufficient condition for the nonlinear effect to be "small-scale" in a Mode III crack problem for a Neo-Hookean material.

1 Introduction

When crack problems are analyzed on the basis of nonlinear theories, such as finite elasticity or deformation theory of plasticity, it is inevitable that nonlinear effects will predominate near a crack-tip, even if the loads are small. The most favorable circumstance in this regard occurs when the loads are so small that the zone of significant nonlinearity lies within the region of validity of the *near-tip* approximation to the global solution of the associated *linearized* crack problem. This situation – called small-scale yielding for crack problems in plasticity – permits simplifications in analysis which are often decisive; see, e.g., Knowles (1977) and Rice (1968).

Insofar as we know, there are no analytical estimates available of the level of load below which nonlinear effects are guaranteed to be small-scale in the above sense. Indeed, even a precise version of the question seems to be lacking. In the present note we formulate and answer such a question for an especially cooperative crack problem; that corresponding to finite anti-plane shear of an infinite medium containing a crack of finite length for an elastic material of Neo-Hookean type. The associated boundary value problem is a linear one for Laplace's equation and thus can be solved globally. Nevertheless, there is a significant nonlinear effect of Kelvin type in the stress field. We give a condition under which this nonlinear response occurs on a small scale near the crack tips.

2 The Crack Problem

In the undeformed state, the elastic body to be considered occupies a cylindrical region whose open cross-section R consists of the exterior of a line segment of length $2l$ (Fig. 1). A right-handed Cartesian coordinate system x_1, x_2, x_3 is used, with the x_1 and x_2 axes as shown in the figure.

The body is to be deformed in finite anti-plane shear with out-of-plane displacement $u = u(x_1, x_2)$. At large distances from the crack, u is to correspond approximately to simple shear with a given amount of shear k: $u \sim kx_2$, where $k \geq 0$ is a constant. The crack faces $x_2 = 0^{\pm}$, $-l < x_1 < l$, are to be traction-free. There are no body forces.

We consider a homogeneous, isotropic, elastic material that is incompressible (anti-plane shear is locally volume-preserving) and of generalized Neo-Hookean type. Materials of this kind are characterized by an elastic potential, or stored energy per unit undeformed volume, W, that depends only on the first fundamental scalar invariant I_1 of the left (or right) Cauchy-Green tensor associated with the deformation: $W = W(I_1)$. The theory of finite anti-plane shear for such materials, as well as its application to the crack problem described above, has been discussed in detail in Knowles (1977). It is shown there that u must be a solution of the following boundary value problem:

$$[2W'(3+|\nabla u|^2)u_{,\alpha}]_{,\alpha}=0 \quad \text{on} \quad R, \tag{1}$$

$$u_{,2}(x_1,0^{\pm})=0, \quad -l<x_1<l, \tag{2}$$

$$u(x_1,x_2)\sim kx_2 \quad \text{as} \quad x_1^2+x_2^2\rightarrow\infty. \tag{3}$$

In addition, u is to be bounded near the tips of the crack. In equation (1), Greek subscripts have the range 1,2, repeated subscripts are summed, and a comma followed by a subscript indicates partial differentiation with respect to the

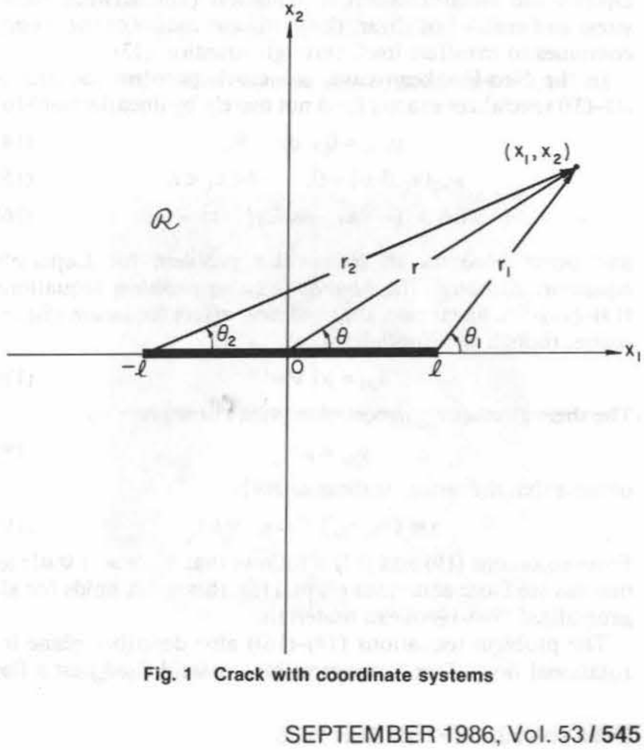

Fig. 1 Crack with coordinate systems

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corresponding Cartesian coordinate; W' stands for the derivative of W with respect to its argument.

In terms of u, the components of the respective true (Cauchy) and nominal (Piola-Kirchhoff) stress tensors τ and σ are given as follows (Knowles, 1977)

$$\tau_{\alpha\beta}=\sigma_{\alpha\beta}=0,\ \tau_{3\alpha}=\tau_{\alpha 3}=\sigma_{3\alpha}=\sigma_{\alpha 3}=2W'(3+|\nabla u|^2)u_{,\alpha},\qquad (4)$$

$$\tau_{33}=2W'(3+|\nabla u|^2)|\nabla u|^2,\ \sigma_{33}=0.\qquad (5)$$

If the crack were absent, so that R coincided with the entire x_1, x_2 plane, conditions (2) would be dropped, and one would have $u = kx_2$ throughout R, corresponding to simple shear. In this event, equations (4) and (5) would reduce to

$$\tau_{\alpha\beta}=\sigma_{\alpha\beta}=0,\quad \tau_{31}=\sigma_{31}=0,\quad \tau_{32}=\sigma_{32}=M(k)k\qquad (6)$$

$$\tau_{33}=M(k)k^2,\quad \sigma_{33}=0\qquad (7)$$

where

$$M(k)=2W'(3+k^2)\qquad (8)$$

is the secant shear modulus at an amount of shear k. It is assumed that $M(k) > 0$ for all k. Note that

$$\mu=M(0)\qquad (9)$$

is the shear modulus for infinitesimal deformation.

In the linear theory of *infinitesimal* anti-plane shear, equations (5) and (7) are replaced by $\tau_{33} = 0$. The non-vanishing of the normal stress τ_{33} is a consequence of nonlinearity and represents an instance of the "Kelvin effect." We note that $\tau_{33} = \mu k^2 + 0(k^4)$ as $k \to 0$ in simple shear.

3 The Neo-Hookean Material

The special W given by

$$W(I_1)=\frac{1}{2}\mu(I_1-3)\qquad (10)$$

corresponds to the Neo-Hookean material, for which equation (8) gives

$$M(k)=\mu \text{ for all } k.\qquad (11)$$

For this material, the components of stress in simple shear follow from equations (6) and (7) as

$$\tau_{\alpha\beta}=0,\quad \tau_{31}=0,\quad \tau_{32}=\mu k,\qquad (12)$$

$$\tau_{33}=\mu k^2.\qquad (13)$$

Despite the linear relation in equation (12) between shear stress and amount of shear, the nonlinear nature of the theory continues to manifest itself through equation (13).

In the Neo-Hookean case, the crack problem (equations (1)-(3)) specializes exactly (and not merely by linearization) to

$$u_{,\alpha\alpha}=0 \quad \text{on} \quad R,\qquad (14)$$

$$u_{,2}(x_1,0\pm)=0,\quad -l<x_1<l,\qquad (15)$$

$$u(x_1,x_2)\sim kx_2 \quad \text{as} \quad x_1^2+x_2^2\to\infty,\qquad (16)$$

and hence becomes an elementary problem for Laplace's equation. Although the *boundary value problem* (equations (14)-(16)) is a linear one, the nonlinear effect (equation (5)) remains, though now specialized to

$$\tau_{33}=\mu|\nabla u|^2.\qquad (17)$$

The shear stresses $\tau_{3\alpha}$ in equation (4) are now given by

$$\tau_{3\alpha}=\mu u_{,\alpha};\qquad (18)$$

observe that the *resultant* shear stress is

$$\tau\equiv(\tau_{3\alpha}\tau_{3\alpha})^{1/2}=\mu|\nabla u|.\qquad (19)$$

From equations (19) and (17) it follows that $\tau_{33}/\tau = |\nabla u|$; as one can see from equations (4) and (5), this result holds for all generalized Neo-Hookean materials.

The problem (equations (14)-(16)) also describes plane irrotational flow of an incompressible, inviscid fluid past a flat plate of width $2l$ at a 90 degree angle of attack; u is the velocity potential and k is the free-stream speed. By symmetry, $\nabla u(0,0\pm) = 0$, corresponding to a stagnation point of the flow.

The global solution of the problem (equations (14)-(16)) may be found in Rice (1968). In terms of the three sets of polar coordinates shown in Fig 1, one has

$$u=k\sqrt{r_1r_2}\sin\frac{\vartheta_1+\vartheta_2}{2},\qquad (20)$$

from which it follows, with the help of equation (18), that

$$\tau_{31}=\frac{\mu kr}{\sqrt{r_1r_2}}\sin\left(\vartheta-\frac{\vartheta_1+\vartheta_2}{2}\right),$$

$$\tau_{32}=\frac{\mu kr}{\sqrt{r_1r_2}}\cos\left(\vartheta-\frac{\vartheta_1+\vartheta_2}{2}\right).\qquad (21)$$

By equations (19) and (21), the resultant shear stress is

$$\tau=\frac{\mu kr}{\sqrt{r_1r_2}}.\qquad (22)$$

From equations (17) and (20), the axial normal stress is found to be

$$\tau_{33}=\frac{\mu k^2r^2}{r_1r_2}.\qquad (23)$$

From equations (20)-(23) one can readily find near-tip approximations for all field quantities. We cite here only that for the resultant shear stress τ near the right crack-tip,

$$\tau\sim\mathring{\tau}\equiv\mu k(l/2r_1)^{1/2} \quad \text{as} \quad r_1\to 0.\qquad (24)$$

For the present special case of the problem (equations (1)-(3)) corresponding to a Neo-Hookean material, the near-tip approximations for all *exact* field quantities (except τ_{33}) and those for their counterparts in the *linearized* problem happen to coincide. This would *not* be the case for generalized Neo-Hookean materials with $M(k) \neq$ constant.

Although according to equations (20) and (21), u and τ_{31} are discontinuous across the crack, equations (22)-(24) show that the resultant shear stress τ, its near-tip approximations $\mathring{\tau}$, and the axial normal stress τ_{33} are continuous everywhere in the x_1, x_2 plane except at the crack-tips, where all three are unbounded. While τ and $\mathring{\tau}$ have the characteristic singularity $0(r_1^{-1/2})$ for linear crack problems, $\tau_{33} = 0(r_1^{-1})$ as $r_1 \to 0$.

4 Scale of the Nonlinear Effect

For the crack problem in the Neo-Hookean case, the ratio τ_{33}/τ provides a natural measure of the size of the local nonlinear effect. If ϵ is the given constant representing a specified error tolerance, $0 < \epsilon < 1$, the elastic field at a given point will be said to be *approximately linear at level* ϵ if $\tau_{33}/\tau<\epsilon$ at that point. Correspondingly, we speak of the ϵ-level *nonlinear* zone N_ϵ as the set of all points (x_1, x_2) such that $\tau_{33}(x_1, x_2)/\tau(x_1, x_2) \geq \epsilon$. From equations (22) and (23), this is the set of points for which

$$\frac{kr}{\sqrt{r_1r_2}}\geq\epsilon.\qquad (25)$$

(We agree to include the crack-tips in N_ϵ.) From the meaning of r_1, r_2, and r, it is clear that – for given k and ϵ – all points sufficiently close to a crack-tip belong to N_ϵ, while all points close enough to the origin do not. (The latter assertion reflects the fact that the origin is a stagnation point of the flow described by equations (14)-(16), so that $\nabla u = 0$ there, whence by equation (17), $\tau_{33} = 0$ as well.) Note that N_ϵ depends on k as well as ϵ; it depends in fact only on the ratio ϵ/k.

Figure 2 shows the ϵ-level nonlinear zone for fixed ϵ and various values of k. From the figure, it may be observed that N_ϵ is disconnected for $k \leq \epsilon$, connected for $k > \epsilon$, bounded for $k < \epsilon$, unbounded for $k \geq \epsilon$. These qualitative features of N_ϵ are as one might expect.

We call the nonlinear effect *enveloping* at level ϵ if N_ϵ is connected. In contrast, if there are two bounded, connected, disjoint sets $N_\epsilon^{(+)}$ and $N_\epsilon^{(-)}$ such that $N_\epsilon = N_\epsilon^{(+)} + N_\epsilon^{(-)}$, while $N_\epsilon^{(+)}$ ($N_\epsilon^{(-)}$) contains the right (left) crack-tip, we say that the nonlinear effect is *contained* at level ϵ. In the present instances, Fig. 2 indicates that the nonlinear effect is contained at level ϵ if $k < \epsilon$, enveloping if $k > \epsilon$. The case $k = \epsilon$ represents the transition.

A second set required for our purposes is the *ϵ-level near-tip linear zone* $L_\epsilon^{(+)}$ associated with the right crack tip. This is the set of points (x_1, x_2) at which the relative error committed by the near-tip approximation $\overset{\circ}{\tau}(x_1, x_2)$ (see equation (24)) to the resultant shear stress $\tau(x_1, x_2)$ is in magnitude at most ϵ: $|\tau(x_1, x_2) - \overset{\circ}{\tau}(x_1, x_2)| / \tau(x_1, x_2) \leq \epsilon$. From equations (22) and (24), $L_\epsilon^{(+)}$ is the set for which

$$\left| 1 - \sqrt{\frac{l}{2}} \, \frac{\sqrt{r_2}}{r} \right| \leq \epsilon. \tag{26}$$

(Again, we include the right crack-tip in $L_\epsilon^{(+)}$.) For given ϵ, all points (x_1, x_2) sufficiently close to the right crack-tip belong to $L_\epsilon^{(+)}$, while all points close enough to the origin do not. One shows easily from equation (26) that $L_\epsilon^{(+)}$ is bounded. It may be observed that $L_\epsilon^{(+)}$ does not depend on k.

Figure 3 shows the ϵ-level near-tip linear zone $L_\epsilon^{(+)}$ for various values of ϵ. In contrast to N_ϵ, $L_\epsilon^{(+)}$ is not symmetric about the x_2 axis, since the approximation $\overset{\circ}{\tau}$ appropriate to the right crack-tip does not have this property. The nature of $L_\epsilon^{(+)}$ is complicated – and unexpected – in two respects, neither of which is important for our purposes. First, even for small error tolerances ϵ such as that for Fig. 3(*a*), $L_\epsilon^{(+)}$ includes points quite remote from the right crack-tip. This is explained by the fact that there is a curve – shown solid in the figure – along which τ and $\overset{\circ}{\tau}$ happen to agree exactly. Second, when larger values of ϵ are considered as in Fig. 3(*c*), $L_\epsilon^{(+)}$ develops a second "cavity" surrounding the left crack-tip. This occurs only for values of ϵ greater than $1-(2\sqrt{2})^{-1} \doteq 0.65$, corresponding to relative error tolerances in excess of 60 percent, which are of little or no interest. In any event, it is only the portion of $L_\epsilon^{(+)}$ near the right crack-tip that is of importance in the analysis to follow.

The ϵ-level near-tip linear zone $L_\epsilon^{(-)}$ associated with the left crack-tip is defined analogously; because of the symmetry of the present problem, $L_\epsilon^{(-)}$ is the image of $L_\epsilon^{(+)}$ under reflection in the x_2 axis.

Inequalities (equations (25) and (26)) defining the respective sets N_ϵ, $L_\epsilon^{(+)}$ are independent of crack length, as is easily shown by the introduction of dimensionless coordinates with $\rho = r/l$, $\rho_\alpha = r_\alpha/l$.

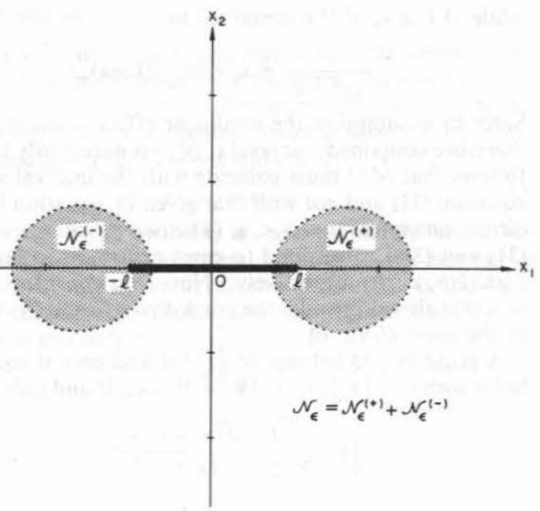

Fig. 2(*a*) **ϵ-level nonlinear zone N_ϵ (shaded), $k = 0.9\,\epsilon$**

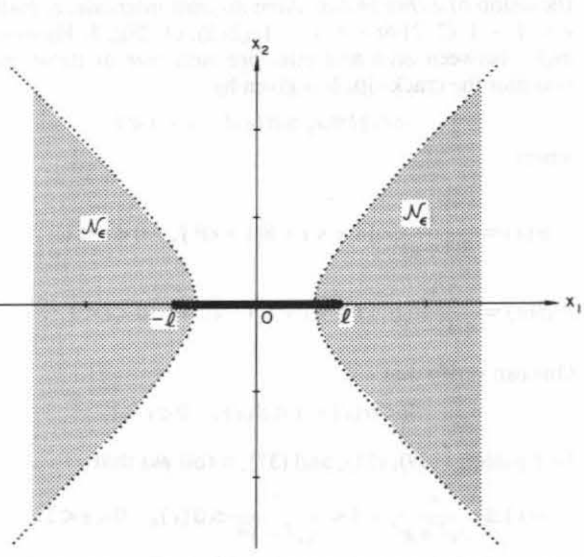

Fig. 2(*b*) **ϵ-level nonlinear zone N_ϵ (shaded), $k = \epsilon$**

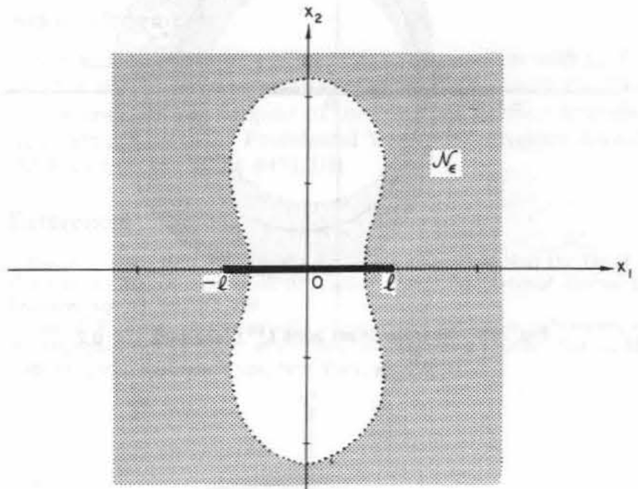

Fig. 2(*c*) **ϵ-level nonlinear zone N_ϵ (shaded), $k = 1.1\,\epsilon$**

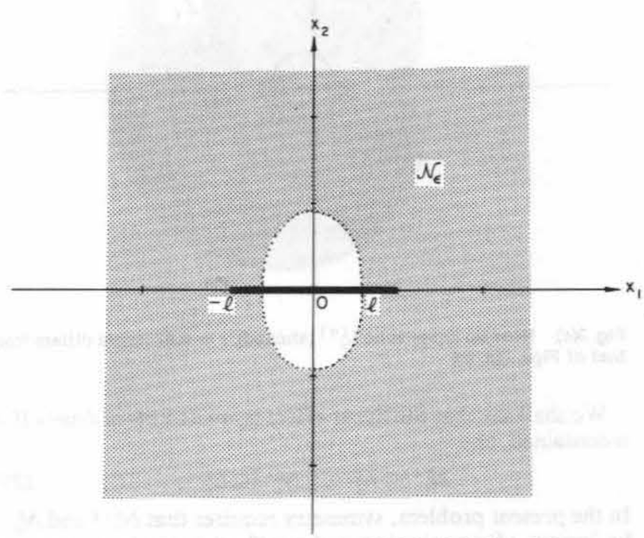

Fig. 2(*d*) **ϵ-level nonlinear zone N_ϵ (shaded), $k = 1.5\,\epsilon$**

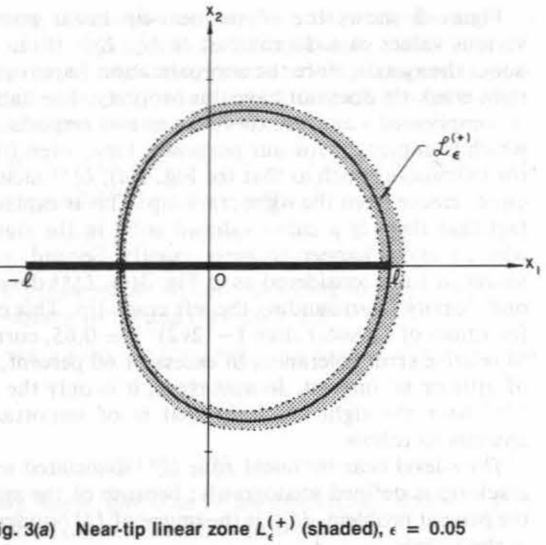

Fig. 3(a) Near-tip linear zone $L_\epsilon^{(+)}$ (shaded), $\epsilon = 0.05$

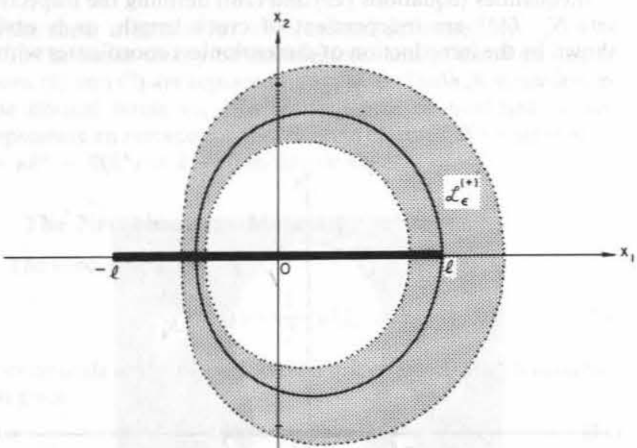

Fig. 3(b) Near-tip linear zone $L_\epsilon^{(+)}$ (shaded), $\epsilon = 0.2$

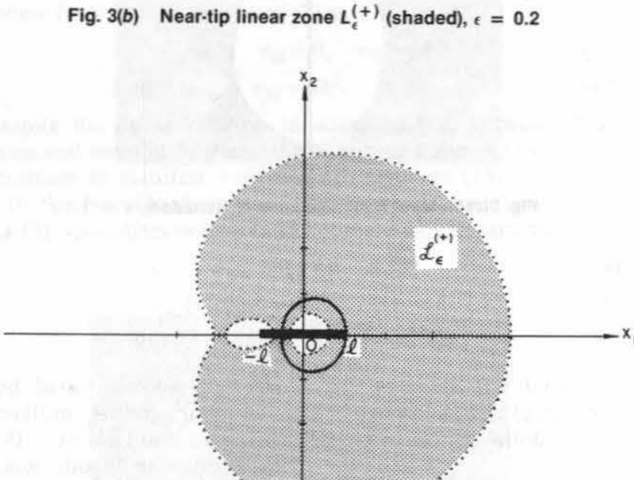

Fig. 3(c) Near-tip linear zone $L_\epsilon^{(+)}$ (shaded), $\epsilon = 0.67$ (scale differs from that of Figs. 3(a, b))

We shall say that the linear effect is *small scale at level* ϵ if it is contained, and

$$N_\epsilon^{(+)} \subseteq L_\epsilon^{(+)}, \quad N_\epsilon^{(-)} \subseteq L_\epsilon^{(-)}. \tag{27}$$

In the present problem, symmetry requires that $N_\epsilon^{(+)}$ and $N_\epsilon^{(-)}$ be images of one another under reflection in the x_2 axis, so that either of the inclusions in equation (27) implies the other. Again by symmetry, $N_\epsilon^{(+)}$ and $N_\epsilon^{(-)}$ must be subsets of the respective open half-planes $x_1 > 0$ and $x_1 < 0$ if the nonlinear effect is contained.

Our objective is to determine, for given ϵ, a condition on the amount of shear at infinity k that is necessary and sufficient for equation (27). Suppose first that the nonlinear effect *is* small scale at level ϵ, so that

$$N_\epsilon^{(+)} \subseteq L_\epsilon^{(+)}. \tag{28}$$

Let $\mathring{N}_\epsilon^{(+)}$ and $\mathring{L}_\epsilon^{(+)}$ designate the respective intersections of $N_\epsilon^{(+)}$ and $L_\epsilon^{(+)}$ with the x_1 axis. By equation (28),

$$\mathring{N}_\epsilon^{(+)} \subseteq \mathring{L}_\epsilon^{(+)}. \tag{29}$$

We now determine $\mathring{N}_\epsilon^{(+)}$ and $\mathring{L}_\epsilon^{(+)}$ explicitly in order to infer from equation (29) a necessary condition on k.

A point $(x_1, 0)$ belongs to $\mathring{N}_\epsilon^{(+)}$ if and only if $x_1 > 0$ and equation (25) holds with $r = x_1$, $r_1 = |x_1 - l|$, $r_2 = x_1 + l$, i.e., if and only if

$$\frac{kx_1}{\sqrt{|x_1^2 - l^2|}} \ge \epsilon. \tag{30}$$

From equation (30) it is easy to show that, if $k < \epsilon$, $\mathring{N}_\epsilon^{(+)}$ coincides with the interval

$$\frac{\epsilon l}{\sqrt{\epsilon^2 + k^2}} \le x_1 \le \frac{\epsilon l}{\sqrt{\epsilon^2 - k^2}}, \quad (k < \epsilon), \tag{31}$$

while, if $k \ge \epsilon$, $\mathring{N}_\epsilon^{(+)}$ is described by

$$\frac{\epsilon l}{\sqrt{\epsilon^2 + k^2}} \le x_1 < \infty, \quad (k \ge \epsilon). \tag{32}$$

Since by assumption the nonlinear effect is small scale – and therefore contained – at level ϵ, $N_\epsilon^{(+)}$ is necessarily bounded. It follows that $\mathring{N}_\epsilon^{(+)}$ must coincide with the interval specified in equation (31) and *not* with that given by equation (32). In addition, one infers that $k < \epsilon$. (The two possibilities (equations (31) and (32)) correspond to cases portrayed in Fig. 2(*a*) and Figs. 2(*b, c, d*), respectively.) Note that the interval in equation (31) always includes the crack-tip $(l, 0)$ and lies to the right of the point $(l/\sqrt{2}, 0)$.

A point $(x_1, 0)$ belongs to $\mathring{L}_\epsilon^{(+)}$ if and only if equation (26) holds with $r = |x_1|$, $r_2 = |x_1 + l|$, i.e., if and only if

$$\left|1 - \sqrt{\frac{l}{2}}\,\frac{\sqrt{|x_1 + l|}}{|x_1|}\right| \le \epsilon. \tag{33}$$

A detailed analysis of equation (33) shows that $\mathring{L}_\epsilon^{(+)}$ consists of the union of *either* two *or* three disjoint intervals, according as $\epsilon < 1 - 1/(2\sqrt{2})$ or $\epsilon \ge 1 - 1/(2\sqrt{2})$; cf. Fig. 3. However, for *any* ϵ between zero and one, precisely *one* of these intervals contains the crack-tip. It is given by

$$\alpha(\epsilon) l \le x_1 \le \beta(\epsilon) l, \quad 0 < \epsilon < 1, \tag{34}$$

where

$$\alpha(\epsilon) = \frac{1}{4(1+\epsilon)^2}\,\{1 + \sqrt{1 + 8(1+\epsilon)^2}\}, \quad 0 < \epsilon < 1, \tag{35}$$

$$\beta(\epsilon) = \frac{1}{4(1-\epsilon)^2}\,\{1 + \sqrt{1 + 8(1-\epsilon)^2}\}, \quad 0 < \epsilon < 1. \tag{36}$$

One can verify that

$$0 < \alpha(\epsilon) < 1 < \beta(\epsilon), \quad 0 < \epsilon < 1. \tag{37}$$

By equations (29), (31), and (37), it follows that

$$\alpha(\epsilon) \le \frac{\epsilon}{\sqrt{\epsilon^2 + k^2}} < 1 < \frac{\epsilon}{\sqrt{\epsilon^2 - k^2}} \le \beta(\epsilon), \quad 0 < \epsilon < 1. \tag{38}$$

It is readily shown that equation (38) implies

$$k \le \min\left\{\frac{\epsilon}{\beta(\epsilon)}\sqrt{\beta^2(\epsilon)-1},\quad \frac{\epsilon}{\alpha(\epsilon)}\sqrt{1-\alpha^2(\epsilon)}\right\},\quad 0<\epsilon<1. \tag{39}$$

Direct calculation shows that the first entry in the braces in equation (39) is smaller than the second one for all ϵ between zero and one, so that equation (39) may be written as

$$k \le k_1(\epsilon),\quad 0<\epsilon<1, \tag{40}$$

where

$$k_1(\epsilon) = \frac{\epsilon}{\beta(\epsilon)}\sqrt{\beta^2(\epsilon)-1},\quad 0<\epsilon<1, \tag{41}$$

with $\beta(\epsilon)$ given by equation (36).

We have shown that *if* the nonlinear effect is small-scale at level ϵ, *then* k must satisfy equation (40) with $k_1(\epsilon)$ given by equations (41) and (36). A simple reversal of the argument leading from equations (29) to (40) shows that equation (40) is *sufficient* for equation (29) as well as necessary. A more troublesome argument, too long to be included here, shows that equation (29) implies (28). It follows that equations (40), (41), and (36) supply a necessary and sufficient condition to be satisfied by the amount of shear at infinity k if the nonlinear effect in the crack problem (equation (14)–(16)) is to be small-scale at level ϵ.

5 Discusssion

For the state of stress (equations (12), (13)) corresponding to simple shear in the absence of the crack, the resultant shear stress is $\tau = (\tau_{3\alpha}\tau_{3\alpha})^{1/2} = \mu k$. For given ϵ, $0 < \epsilon < 1$, the state of simple shear is approximately linear at level ϵ if $\tau_{33}/\tau < \epsilon$, or – by equations (12), (13) – if and only if

$$k < k_o(\epsilon) \equiv \epsilon. \tag{42}$$

Since for the crack problem an approximately linear state throughout the body is unattainable for any ϵ, the best one can do is secure a state in which the nonlinear effect is small scale at level ϵ. According to the result of the preceding section, this will occur if and only if

$$k \le k_1(\epsilon), \tag{43}$$

where $k_1(\epsilon)$ is given by equations (41) and (36). For small ϵ, one can show that

$$k_1(\epsilon) \sim 2\sqrt{2/3}\,\epsilon^{3/2} \quad \text{as} \quad \epsilon \to 0. \tag{44}$$

Comparison of equations (44), (42) shows that, for small error tolerances ϵ, the restriction (equation (43)) on k required for the nonlinear effect to be small scale at level ϵ is somewhat more severe than that in equation (42) which is necessary to secure an approximately linear state of simple shear at level ϵ in the absence of the crack.

If the present problem were modified to allow the crack to make an angle other than 90 deg with the x_2 axis, the general notions introduced here concerning the nonlinear zone, confined and small-scale nonlinear effects remain applicable, but now in general $N_\epsilon^{(+)}$ and $N_\epsilon^{(-)}$ are no longer reflections of one another and the two inclusions in equation (27) are no longer equivalent. Thus the nonlinear effect might be small scale at one crack-tip but not at the other.

If the elastic potential $W(I_1)$ is not that for a Neo-Hookean material, the boundary value problem (1)–(3) no longer permits the analytical construction of a global solution, and the methods used here are thus not applicable. Moreover, the definition of the nonlinear zone would require modification, since the presence of a nonvanishing axial normal stress τ_{33} is no longer the only nonlinear effect present. These remarks also apply to situations with more complicated kinematics, such as Mode I or II crack problems in finite plane strain.

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References

Knowles, J. K., 1977, "The Finite Anti-plane Shear Field Near the Tip of a Crack for a Class of Incompressible Elastic Solids," *International Journal of Fracture*, Vol. 13, pp. 611–639.

Rice, J. R., 1968, "Mathematical Analysis in the Mechanics of Fracture, in the Mechanics of Fracture," in *Fracture: an Advanced Treatise*, Vol. 2, H. Liebowitz, ed., Academic Press, New York, pp. 191–311.